



SYNCHRONIZATION BEHAVIOR OF RESTRICTED THREE BODY PROBLEM WHEN BIGGER PRIMARY IS AN UNIFORM CIRCULAR DISC

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Abstract: In this article author discuss the synchronization behaviour of the planar restricted three body problem when the bigger primary is an uniform circular disc evolving from deferent initial conditions using active control technique based on the Lyapunov-stability theory and Routh-Hurwitz criteria. Numerical simulations are performed to plot phase portraits, time series analysis graphs of the master system and the slave system which further illustrate the effectiveness of the proposed control techniques.

Keywords Space dynamics, restricted three body problem, Synchronization, Lyapunov stability theory and Routh- Hurwitz criteria.

1. INTRODUCTION

In the last two decades considerable research has been done in non-linear dynamical systems and their various properties. One of the most important properties is synchronization which is an important topic in the nonlinear dynamics. Pec-ora and Carroll [1] introduced a method to synchronize two identical chaotic systems with deferent initial conditions in (1990) and they demonstrated that chaotic synchronization could be achieved by driving or replacing one of the variables of a chaotic system with a variable of another similar chaotic device. Many methods and techniques for chaos control and synchronization of various chaotic systems have been developed, such as non linear feedback control [2] , OGY approach [3], sliding mode control [4], adaptive synchronization [5] , anti synchronization method [6], and so on. The synchronization problem via nonlinear control scheme is studied by Amir Abbas Emadzadeh, and Mohammad Haeri [8], M. Mossa Al-sawalha, M.S.M. Noorani in [9], Chen and Han [10], Chen [11], Ju H. Park [12] etc. Chaos synchronization using active control was proposed by Bai and Lonngren [13] in (1997) and has recently been widely accepted as an efficient technique for the synchronization of chaotic systems, because it can be used to synchronize non-identical systems as well; a feature that gives it an advantage over other synchronization methods. This method has been applied to many practical systems such as spatiotemporal dynamical systems (Codreanu [14]), the Rikitake two-disc dynamo-a geographical systems (Vincent [15]), Non-linear Bloch equations modeling "jerk" equation and R. C. L shunted Josephson junctions (Ucar et al. [16,17]), etc.

Extensive research work has been devoted to address the circular restricted three body problem in the field of astronomy and space dynamics. A numbers of good research papers have investigated the circular restricted three body problem such as the Euler [18], Hill [19], Poincare [20], Lagrange [21], Deprit [22], Hadjidemetriou [23], Bhatnagar [24,25], Sahoo and Ishwar [26] Sharma et al. [27,28], and many others. These studies focus on the analytical, qualitative and numerical studies of the problem. A detailed analysis of this problem is illustrated in the work of American mathematician Szebehely [29].

In (2017) Mohd. Arif and Ravi kumar sagar [30] have discussed the planer restricted three body problem by taken in to consideration the bigger primary is an uniform circular disc. Khan and Shahzad [31] investigated the synchronization behaviour of the two identical circular restricted three body problem influenced by radiation evolving from different initial conditions via the active control. In (2013) Khan and Tripathi [32] have investigated the synchronization behavior of a restricted three body problem under the effect of radiation pressure. In an another paper the Complete synchronization, anti-synchronization and hybrid synchronization of two identical parabolic restricted three body problem have been studied by Khan and Rimpi pal [33].

Being motivated by the above discussion, in this article we have discussed the complete synchronization behavior of the planar restricted three body problem when the bigger primary is an uniform circular disc evolving from deferent initial conditions using active control technique based on the Lyapunov-stability theory and Routh-Hurwitz criteria. It has been observed that the system is chaotic for some values of parameter. Hence the slave chaotic system completely traces the dynamics of the master system in the course of time. The paper is organized as follows.

In section 2 we have discussed the complete synchronization behavior of the problem via active control technique. Finally, we conclude the paper in section 3.

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2. COMPLETE SYNCHRONIZATION

The equation of motion for the restricted three body problem when the bigger primary is uniform circular disc in a dimensionless, barycentric, pulsating rotating, co-ordinate system are as follows.

$$\ddot{x} - 2n \dot{y} = U_x \tag{1}$$

$$\ddot{y} + 2n\dot{x} = U_y \tag{2}$$

Where

$$U = \frac{n^2(x^2+y^2)}{2} + \frac{\mu}{r_2} - 2G\sigma[(a+r_1)E(k) + (a-r_1)K(k)] \tag{3}$$

$$r_1^2 = (x-\mu)^2 + y^2, \quad r_2^2 = (x+1-\mu)^2 + y^2$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2 \sin^2 \psi} d\psi, \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2 \sin^2 \psi}}, \quad k^2 = \frac{4ar}{(a+r)^2} < 1$$

Let

$$x = x_1, \quad \dot{x} = x_2, \quad y = x_3, \quad \dot{y} = x_4$$

Then the equation (1) and (2) can be written as:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2nx_4 + n^2x_1 + A_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -2nx_2 + n^2x_3 + B_1 \end{aligned} \right\} \tag{4}$$

Where

$$A_1 = -\frac{\mu}{r_2^3}(x_1 + \mu - 1) + \frac{2\sigma(x_1 - \mu)}{r_1} \left[(E(k) - K(k)) \left(1 + \frac{4a(a-r_1)}{k(a+r_1)^2} \right) + \frac{4a(a-r_1)^2}{(a+r_1)^3} \left(\frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \right) \right]$$

$$B_1 = -\frac{\mu}{r_2^3}x_3 + \frac{2\sigma x_3}{r_1} \left[(E(k) - K(k)) \left(1 + \frac{4a(a-r_1)}{k(a+r_1)^2} \right) + \frac{4a(a-r_1)^2}{(a+r_1)^3} \left(\frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \right) \right]$$

$$r_1^2 = (x_1 - \mu)^2 + x_3^2, \quad r_2^2 = (x_1 + 1 - \mu)^2 + x_3^2.$$

Corresponding to master system (4), the identical slave system is defined as:

$$\left. \begin{aligned} \dot{y}_1 &= y_2 + u_1(t) \\ \dot{y}_2 &= 2ny_4 + n^2y_1 + A_2 + u_2(t) \\ \dot{y}_3 &= y_4 + u_3(t) \\ \dot{y}_4 &= -2ny_2 + n^2y_3 + B_2 + u_4(t) \end{aligned} \right\} \tag{5}$$

Where

$$A_2 = -\frac{\mu}{r_{21}^3}(y_1 + \mu - 1) + \frac{2\sigma(y_1 - \mu)}{r_{11}} \left[(E(k) - K(k)) \left(1 + \frac{4a(a-r_{11})}{k(a+r_{11})^2} \right) + \frac{4a(a-r_{11})^2}{(a+r_{11})^3} \left(\frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \right) \right]$$

$$B_2 = -\frac{\mu}{r_{21}^3}y_3 + \frac{2\sigma y_3}{r_{11}} \left[(E(k) - K(k)) \left(1 + \frac{4a(a-r_{11})}{k(a+r_{11})^2} \right) + \frac{4a(a-r_{11})^2}{(a+r_{11})^3} \left(\frac{E(k) - (1-k^2)K(k)}{k(1-k^2)} \right) \right]$$

$$r_{11}^2 = (y_1 - \mu)^2 + y_3^2, \quad r_{21}^2 = (y_1 + 1 - \mu)^2 + y_3^2$$

And $u_i(t); i=1,2,3,4$ are control functions to be determined.

Let $e_i = y_i - x_i; i=1,2,3,4$ be the synchronization errors. From (4) and (5), we obtain the error dynamics as follows:

$$\left. \begin{aligned} \dot{e}_1 &= e_2 + u_1(t) \\ \dot{e}_2 &= 2ne_4 + n^2e_1 + A_2 - A_1 + u_2(t) \\ \dot{e}_3 &= e_4 + u_3(t) \\ \dot{e}_4 &= -2e_2 + n^2e_3 + B_2 - B_1 + u_4(t) \end{aligned} \right\} \tag{6}$$

This error system to be controlled is a linear system with control functions. Thus, let us redefine the control functions so that the terms in (6) which cannot be expressed as linear terms in e_i 's are eliminated :

$$\left. \begin{aligned} u_1(t) &= v_1(t) \\ u_2(t) &= -A_2 + A_1 + v_2(t) \\ u_3(t) &= v_3(t) \end{aligned} \right\} \tag{7}$$

$$u_4(t) = -B_2 + B_1 + v_4(t)$$

The new error system can be expressed as:

$$\left. \begin{aligned} \dot{e}_1 &= e_2 + v_1(t) \\ \dot{e}_2 &= 2ne_4 + n^2e_1 + v_2(t) \\ \dot{e}_3 &= e_4 + v_3(t) \\ \dot{e}_4 &= -2ne_2 + n^2e_3 + v_4(t) \end{aligned} \right\} \quad (8)$$

The above error system to be controlled is a linear system with a control input $v_i(t)$ ($i = 1, \dots, 4$) as function of the error states e_i ($i = 1, \dots, 4$). As long as these feedbacks stabilize the system e_i ($i = 1, \dots, 4$) converge to zero as time t tends to infinity. This implies that master and the slave system are synchronized with active control. We choose.

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \\ v_4(t) \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \quad (9)$$

Here A is a 4×4 coefficient matrix to be determined. As per Lyapunov stability theory and Routh-Hurwitz criterion, in order to make the closed loop system (9) stable, proper choice of elements of A has to be made so that the system (9) must have all eigen values with negative real parts. Choosing

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -n^2 & -1 & 0 & -2n \\ 0 & 0 & -1 & -1 \\ 0 & 2n & -n^2 & -1 \end{bmatrix} \quad (10)$$

and, defining a matrix B as

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = B \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

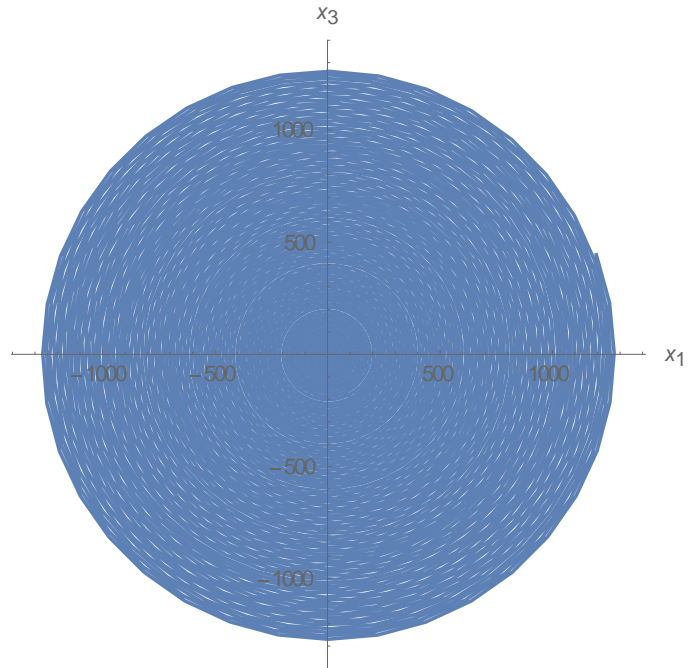
Where B is

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (11)$$

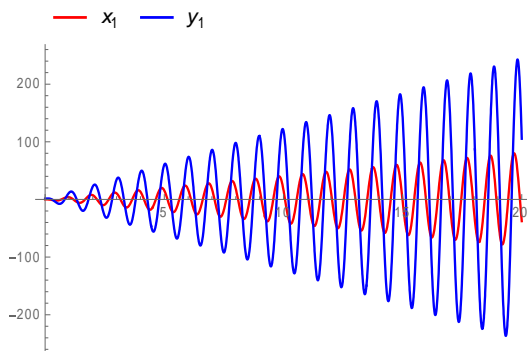
Clearly, B has eigen values with negative real parts. This implies $\lim_{t \rightarrow \infty} |e_i| = 0$; $i = 1, 2, 3, 4$ and hence, complete synchronization is achieved between the master and slave systems. Time Series Analysis graphs of the above are shown next to each via figures 2 and 9.

3. NUMERICAL SIMULATION

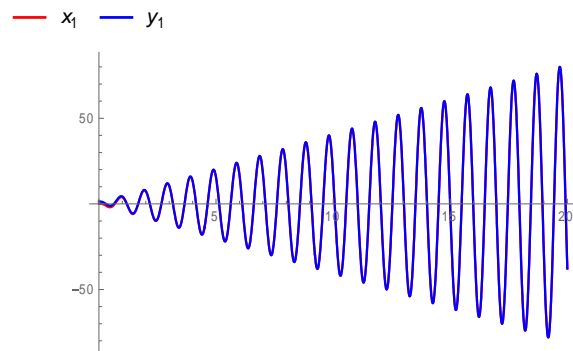
We select the parameters $\mu = .00001$ and radius of the disc $\alpha = .03$, the state orbits of the chaotic system are shown in Figure 1, with the different initial conditions $[x_1(0) = 1.9, x_2(0) = 1.6, x_3(0) = 1.8, x_4(0) = 3.02]$ for master systems. Simulation results for uncoupled system are presented in figures. 2,4,6 and 8 and that of controlled system are shown in figures. 3,5,7 and 9 respectively.



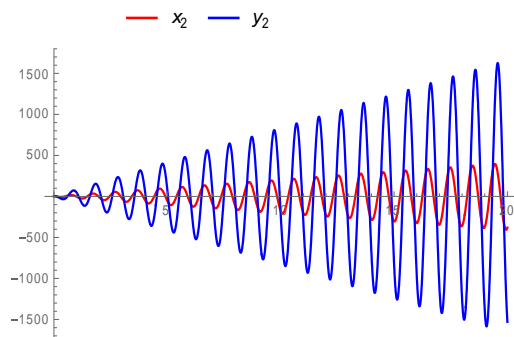
Fig(1)



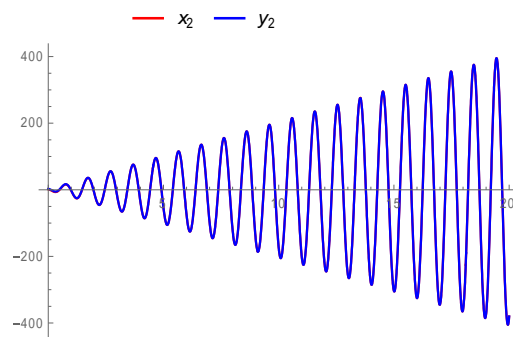
Fig(2)



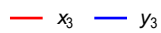
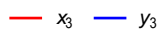
Fig(3)

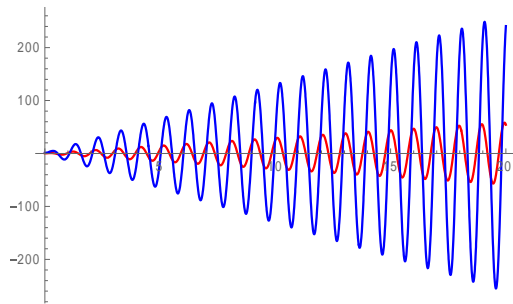


Fig(4)

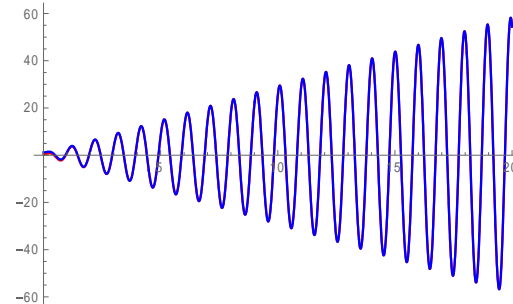


Fig(5)

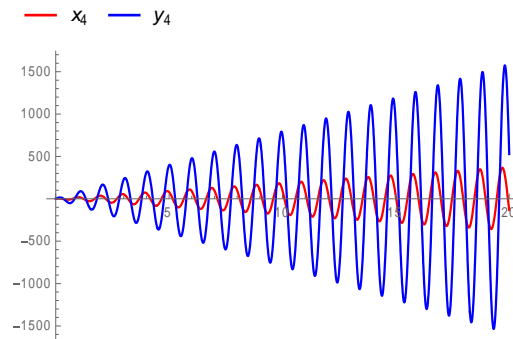




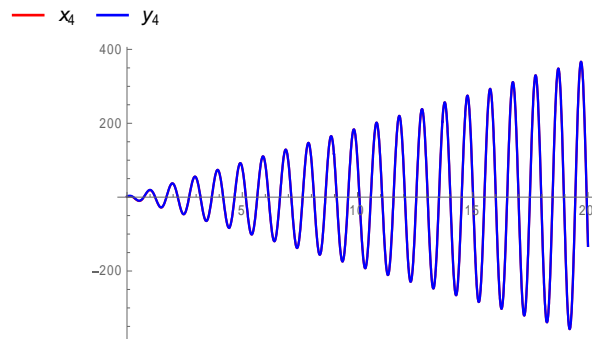
Fig(6)



Fig(7)



Fig(8)



Fig(9)

4. CONCLUSION

In this article authors investigated the complete synchronization behavior of the planar restricted three body problem when the bigger primary is an uniform circular disc using active control technique based on the Lyapunov-stability theory and Routh-Hurwitz criteria,. Here two systems (master and slave) are completely synchronized evolving from different initial conditions.

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